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NONLINEAR WAVE EQUATION WITH VANISHING POTENTIAL

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Abstract. We study the Cauchy problem for $u_{tt} - \Delta u + V(x)u^5 = 0$ in 3–dimensional case. The function $V(x)$ is positive and regular, in particular we are interested in the case $V(x) = 0$ in some points. We look for the global classical solution of this equation under a suitable hypothesis on the initial energy.

1. Introduction. This work deals with the existence of the global classical solution for the Cauchy problem related to the equation

$$\Box\, u(x,t) = -V(x)\Phi(u)$$

with large initial data. Here Φ is a smooth positive function with polynomial growth at infinity and $V(x)$ is a $\mathcal{C}^2$ positive function. Since the support of the solution becomes large for large t, the growth of $V(x)$ could influence the existence of the solution; this makes the previous equation different from the classical one, in which $V(x)$ is constant.

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A large amount of work has been devoted to the study of the regularity theory for the case with constant potential, i.e.

$$\Box\, u(x,t) = -\lambda |u|^{p-1}u, \qquad \lambda > 0,\, p \geq 1.$$

In particular, we recall that $p = \dfrac{n+2}{n-2}$ is the critical exponent for the global existence of classical solutions. Many papers treat the sub-critical case in which the existence of a unique regular solution of the previous equation, with regular Cauchy data, is known; if $p = \dfrac{n+2}{n-2}$ the same has been proved only for $3 \leq n \leq 7$. A proof of this can be found in Shatah–Struwe [7]. No result is known on the existence of the global classical solution in the super–critical case.
A crucial step toward the critical result was made by Rauch in [6]: he observed that in the case $n = 3$, $p = 5$, to obtain global existence it suffices to have a particular upper bound for the initial energy. His technique was based on Kirchhoff representation formula.

We briefly recall the idea used in these papers to establish that a local solution $u : [0,T[\times\mathbb{R}^n \to \mathbb{R}$ is a global solution. One supposes that the solution blows up, namely there exists $X \in \mathbb{R}^n$ and a sequence $(x_n, t_n) \to (X,T)$ such that $\lim_n |u(x_n,t_n)| = +\infty$. On the other hand, one tries to find a neighborhood for (X,T) where u is bounded. This contradiction gives the global existence.

If $V(x) > 0$, locally it behaves like a positive constant, Shatah–Struwe result still holds (cf. [4]). On the contrary, if $V(x)$ vanishes the zero of the potential could compensate the blow up of the solution. In particular the Shatah–Struwe technique is not directly available in force of the following two problems:

(1) the lack of a Pohozaev type identity;
(2) the lack of a multiplicative inequality in Besov spaces.

On the other hand, if $V(x) = 0$ in $\overline{x}$, the equation at that point reduces to the linear homogeneous wave equation; hence it is natural to think that the global existence result is still valid.

In this paper, to overcome the previous difficulties, we come back to Rauch's approach. We treat the 3-dimensional critical case, i.e. $\Phi(u) = u^5$.

The plan of the work is the following: in Section 2 we fix the notations and state some standard results useful for our aim; in Section 3 (Theorem 3.1) we prove that if $V(x) = |x - \overline{x}|^{\alpha}$, there exists $\varepsilon_0(\alpha)$ such that if the initial energy is not greater than $\varepsilon_0(\alpha)$, the problem has a unique global solution; we find that $\varepsilon_0(\alpha)$ is increasing in α; in Section 4 we determine other conditions on $V(x)$ which implies the global existence results. Finally we investigate the possible distribution for the zeros of the potential.

2. Notations and known results. Let us consider the Cauchy problem

$$\Box\, u(x,t) = -V(x)\Phi(u) \tag{2.1}$$

$$u(0,x) = f(x)$$
$$u_t(0,x) = g(x), \tag{2.2}$$

where V, Φ satisfy

(i) $V \in \mathcal{C}^2(\mathbb{R}^n), \quad V(x) \geq 0;$

(ii) $\Phi \in \mathcal{C}^2(\mathbb{R}),\ \int_0^u \Phi(t)dt \geq 0,\ \Phi(0) = 0, \quad \Phi(u) = |u|^{p-1}u \textit{ if } |u| \geq C > 0.$

Since we want to prove the boundedness of u and $\Phi(u) = |u|^{p-1}u$ for large $|u|$, we can assume without restriction that, instead of (2.1), u solves the simpler equation

$$\Box\, u = -V(x)|u|^{p-1}u. \tag{2.3}$$

We recall that for this equation there is finite speed of propagation equal to one and there is formal conservation law for the energy

$$E[u](t) = \frac{1}{2}\int_{\mathbb{R}^n} \left(|u_t(x,t)|^2 + |\nabla_x u(x,t)|^2\right)\,\mathrm{d}x + \frac{1}{p+1}\int_{\mathbb{R}^n} V(x)|u(x,t)|^{p+1}\,\mathrm{d}x, \tag{2.4}$$

that is $E[u](t) = E[u](0)$ for each $t \in \mathbb{R}$. We'll put $E[u](0) =: E_0$.

Using these properties one can prove the local existence result (cf. [5, Theorem 4.3]):

Theorem 2.1. *Let $s > n/2$; for any $(f,g) \in H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n)$ there exists $T > 0$ and a unique strong s-regular solution $u(x,t) : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ for* (2.1), (2.2)*, that is $u \in \mathcal{C}([0,T], H^s(\mathbb{R}^n))$, $u_t \in \mathcal{C}([0,T], H^{s-1}(\mathbb{R}^n))$ and $u_{tt} \in \mathcal{C}([0,T], H^{s-2}(\mathbb{R}^n))$.*
If $f \in \mathcal{C}^3(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$, $g \in \mathcal{C}^2(\mathbb{R}^n) \cap H^{s-1}(\mathbb{R}^n)$ the unique local solution of (2.1), (2.2) *is a classical solution, i.e. $u \in \mathcal{C}^2(\mathbb{R}^n \times [0,T])$.*
Moreover, if (f,g) have compact support, then also $u(\,\cdot\,,t)$ has compact support for each $t \in [0,T]$.

Having in mind the finite speed of propagation, we consider the backward cone with vertex $\overline{z} = (\overline{x},\overline{t}) \in \mathbb{R}^n \times \mathbb{R}$:

$$K(\overline{z}) := \{z = (x,t) \mid t \leq \overline{t} \quad |x - \overline{x}| \leq \overline{t} - t\}.$$

We put

$$\begin{aligned}
&K_S^T(\overline{z}) := \{(x,t) \in K(\overline{z}) \mid S \le t \le T\}, \quad K_S(\overline{z}) := K_S^{\overline{t}}(\overline{z}),\\
&M(\overline{z}) := \{z = (x,t) \mid t \le \overline{t} \quad |x-\overline{x}| = \overline{t} - t\},\\
&M_S^T(\overline{z}) := \{(x,t) \in M(\overline{z}) \mid S \le t \le T\}, \quad M_S(\overline{z}) := M_S^{\overline{t}}(\overline{z}),\\
&D(t,\overline{z}) := \{x \in \mathbb{R}^n \mid (x,t) \in K(\overline{z})\}.
\end{aligned}$$

In what follows we denote by $B_r(x)$ the ball having centre in x and radius $r > 0$. Moreover we define the local energy

$$E(u, D(t,\overline{z})) := \int_{D(t,\overline{z})} \frac{1}{2}|u_t|^2 + \frac{1}{2}|\nabla_x u|^2 + \frac{V(x)}{p+1}|u|^{p+1}\,\mathrm{d}x \tag{2.5}$$

and we state a crucial information for the proof of our theorem, namely the flux conservation law: for $S \le T \le \overline{t}$

$$\begin{aligned}
&E(u, D(S,\overline{z})) - E(u, D(T,\overline{z})) =\\
&= \frac{1}{\sqrt{2}} \int_{M_S^T(\overline{z})} \frac{1}{2}\left|\frac{x-\overline{x}}{|x-\overline{x}|}u_t - \nabla_x u\right|^2 + V(x)\frac{|u|^{p+1}}{p+1}\,\mathrm{d}\omega.
\end{aligned} \tag{2.6}$$

This is obtained integrating on $K_S^T(\overline{z})$ the identity

$$\partial_t\left(\frac{1}{2}|u_t|^2 + \frac{1}{2}|\nabla_x u|^2 + \frac{V(x)}{p+1}|u|^{p+1}\,\mathrm{d}x\right) = \operatorname{div}(u_t\,\nabla_x u).$$

Using the coordinates $(x, \overline{t} - |x-\overline{x}|)$ the flux conservation law can be written as

$$\begin{aligned}
&E(u, D(S,\overline{z})) - E(u, D(T,\overline{z})) =\\
&= \int_{D(S,\overline{z})\setminus D(T,\overline{z})} \frac{1}{2}|\nabla u(x, \overline{t} - |x-\overline{x}|)|^2 + \frac{V(x)}{p+1}|u|^{p+1}(x, \overline{t} - |x-\overline{x}|)\,\mathrm{d}x.
\end{aligned} \tag{2.7}$$

3. The case $\boldsymbol{V(x) = |x-\overline{x}|^\alpha}$. In what follows, we consider only the 3-dimensional case in which Kirchhoff formula (see (3.6)) is valid.

By the aid of this representation formula, in the case $p < 5$, one can find a unique global solution to the Problem (2.1), (2.2) for each $\mathcal{C}^2$ positive potential. This can be proved following the classical Jörgens argument (see [6]).

Instead, in the critical case $p = 5$ to obtain the global solution we need further information on the size of the initial data.

In this section we restrict our attention to the potential $V(x) = |x-\overline{x}|^\alpha$ and prove the following theorem:

Theorem 3.1. *Let $\alpha \in \mathbb{R}$, $\alpha = 0$ or $\alpha \ge 2$. Let us consider the Cauchy* *problem*

$$\square\, u(x,t) = -|x-\overline{x}|^{\alpha} u^5 \tag{3.1}$$

$$u(0,x) = f(x)$$
$$u_t(0,x) = g(x) \tag{3.2}$$

with $x \in \mathbb{R}^3$, $f \in \mathcal{C}^3$ $g \in \mathcal{C}^2$. *There exists* $\varepsilon_0 = \varepsilon_0(\alpha)$, *such that if the initial energy* E_0 *satisfies*

$$E_0 \le \varepsilon_0(\alpha), \tag{3.3}$$

then the Problem (3.1), (3.2) *admits a unique global classical solution.*
Moreover $\varepsilon_0(\alpha)$ *is an increasing function in* α.

This theorem will be proved by the aid of a suitable Hardy inequality.

Lemma 3.1. *Let* $\overline{x} \in \mathbb{R}^n$, $\alpha > 2-n$, $n \ge 3$. *For all* $\omega \in \mathcal{C}_0^1(\mathbb{R}^n)$ *one has*

$$\int_{\mathbb{R}^n} |x-\overline{x}|^{\alpha-2} |\omega(x)|^2 \,\mathrm{d}x \le \frac{4}{(\alpha-2+n)^2} \int_{\mathbb{R}^n} |x-\overline{x}|^{\alpha} |\nabla\omega(x)|^2 \,\mathrm{d}x. \tag{3.4}$$

This lemma follows from Theorem 14.4 in [2]; a direct proof can be found in [3]. For $\alpha = 0$ and $n = 3$ this is used by Rauch in [6].
In what follows we put $K_{\alpha,n} = 4/(\alpha-2+n)^2$ and $K_\alpha = K_{\alpha,3}$.

For the proof of our theorem we have to localize the previous inequality.

Lemma 3.2. *Let be* $R > 0$, $\alpha \ge 0$, $\overline{x} \in \mathbb{R}^n$ *and* $\omega \in \mathcal{C}^1(B_{2R}(\overline{x}))$. *The following inequality holds:*

$$\int_{B_{2R}(\overline{x})} |x-\overline{x}|^{\alpha-2} \omega^2 \,\mathrm{d}x \le$$
$$\le K_{\alpha,n} \int_{B_{2R}(\overline{x})} |x-\overline{x}|^{\alpha} |\nabla\omega|^2 \,\mathrm{d}x + \frac{CK_{\alpha,n}+1}{R^2} \int_{B_{2R}(\overline{x})} |x-\overline{x}|^{\alpha} \omega^2 \,\mathrm{d}x. \tag{3.5}$$

P r o o f. Let us fix a cut function $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^3)$, $0 \le \eta \le 1$, such that $\eta = 1$ if $|x-\overline{x}| \le R$, $\eta = 0$ if $|x-\overline{x}| \ge 2R$; moreover we require $|\nabla\eta| \le C/R$. Hence

$$\int_{|x-\overline{x}|\le 2R} |x-\overline{x}|^{\alpha-2} \omega^2 \,\mathrm{d}x \le$$
$$\le \int_{|x-\overline{x}|\le 2R} |x-\overline{x}|^{\alpha-2} (\eta\omega)^2 \,\mathrm{d}x + \int_{R\le|x-\overline{x}|\le 2R} |x-\overline{x}|^{\alpha-2} \omega^2 \,\mathrm{d}x \le$$
$$\le K_{\alpha,n} \int_{\mathbb{R}^n} |x-\overline{x}|^{\alpha} |\nabla(\eta\omega)|^2 \,\mathrm{d}x + \frac{1}{R^2} \int_{R\le|x-\overline{x}|\le 2R} |x-\overline{x}|^{\alpha} \omega^2 \,\mathrm{d}x.$$

In force of our choice of η we get (3.5). $\square$

Now we are in position to establish our result.

Proof of Theorem 3.1. Let be $u(x,t) : \mathbb{R}^3 \times [0,\overline{t}[\to \mathbb{R}$ the maximal local classical solution of (3.1), (3.2). Let be $\overline{z} = (\overline{X}, \overline{t})$ the eventual point in which the blow up occurs.

If $\overline{X} \neq \overline{x}$, there exists $\delta > 0$ such that $|x - \overline{x}|^\alpha > 0$ in $B_\delta(\overline{X})$. In [4] is shown that there exist $t_0 \geq \overline{t} - \delta$, $\varepsilon > 0$ such that the solution is bounded in $K_{t_0}(\overline{X}, \overline{t} + \varepsilon)$. This means that $\overline{z}$ is not a blow up point.

Let us fix our attention to the point $\overline{z} = (\overline{x}, \overline{t})$. Let be $t_0 \leq \overline{t}$. For any $(x,t) \in K_{t_0}(\overline{z}) \setminus \{\overline{z}\}$, Kirchhoff formula gives

$$u(x,t) = \underline{u}(x,t) + \frac{1}{4\pi} \int_{t_0}^{t} (t-\sigma)^{-1} \int_{|x-\xi|=t-\sigma} |\xi - \overline{x}|^\alpha u^5(\xi,\sigma)\, \mathrm{d}\xi\, \mathrm{d}\sigma, \tag{3.6}$$

where $\underline{u}(x,t)$ is the solution of the linear homogeneous equation with the same initial data. From this we obtain:

$$\begin{aligned} &\|u(x,t) - \underline{u}(x,t)\|_{L^\infty(\mathbb{R}^3 \times [t_0, \overline{t}])} \leq \\ &\leq \frac{1}{4\pi} \|u(x,t)\|_{L^\infty(\mathbb{R}^3 \times [t_0,\overline{t}])} \sup_{(x,t) \in \mathbb{R}^3 \times [t_0,\overline{t}]} \int_{t_0}^{t} (t-\sigma)^{-1} \int_{|x-\xi|=t-\sigma} |\xi - \overline{x}|^\alpha u^4\, \mathrm{d}\xi\, \mathrm{d}\sigma. \end{aligned} \tag{3.7}$$

When $\overline{t} < +\infty$ we'll prove

$$\int_{t_0}^{t} (t-\sigma)^{-1} \int_{|x-\xi|=t-\sigma} |\xi - \overline{x}|^\alpha u^4\, \mathrm{d}\xi\, \mathrm{d}\sigma \leq 2\pi \tag{3.8}$$

for each t_0 such that $|\overline{t} - t_0| \leq 1$.

This suffices to establish the existence of global solution; in fact from (3.6) and (3.8) we have $u \in L^\infty(K_{t_0}(\overline{z}))$, while if u blows up in $(\overline{x}, \overline{t})$ we could choose a sequence $\{(x_n, t_n)\}$ in $K_{t_0}(\overline{z})$ such that $(x_n, t_n) \to (0, \overline{t})$ and $|u(x_n, t_n)| \to +\infty$. This contradiction necessary gives $\overline{t} = \infty$.

Now we prove (3.8). Let $z = (x,t) \in K_{t_0}(\overline{z}) \setminus \{\overline{z}\}$, we note that

$$\int_{t_0}^{t} (t-\sigma)^{-1} \int_{|x-\xi|=t-\sigma} |\xi - \overline{x}|^\alpha u^4\, \mathrm{d}\xi\, \mathrm{d}\sigma \leq \int_{M_{t_0}^t(\overline{z})} |\xi - \overline{x}|^{\alpha-1} u^4(\xi,\sigma)\, \mathrm{d}\omega.$$

By using coordinates $(x, t - |x - \overline{x}|)$ on $M_{t_0}^t(\overline{z})$, and denoting by $v(x) = u(x, t - |x - \overline{x}|)$ we find:

$$\int_{M_{t_0}^t(\overline{z})} |\xi - \overline{x}|^{\alpha-1} u^4(\xi,\sigma)\, \mathrm{d}\omega = \sqrt{2} \int_{D(t_0,\overline{z}) \setminus D(t,\overline{z})} |x - \overline{x}|^{\alpha-1} v^4(x)\, \mathrm{d}x \leq$$

$$\leq \sqrt{2}\left(\int_{D(t_0,\overline{z})} |x-\overline{x}|^{\alpha} v^6(x)\,\mathrm{d}x\right)^{1/2} \left(\int_{D(t_0,\overline{z})} |x-\overline{x}|^{\alpha-2} v^2(x)\,\mathrm{d}x\right)^{1/2}. \tag{3.9}$$

From flux conservation law (2.7) it follows that $E(u, D(t,\overline{z}))$ is a decreasing function; hence

$$\int_{D(t_0,\overline{z})} |x-\overline{x}|^{\alpha} v^6\,\mathrm{d}x \leq 6E_0.$$

In order to estimate the second factor in (3.9) we use Lemma 3.2. We get

$$\int_{D(t_0,\overline{z})} |x-\overline{x}|^{\alpha-2} v^2\,\mathrm{d}x \leq$$

$$\leq K_\alpha \int_{D(t_0,\overline{z})} |x-\overline{x}|^{\alpha} |\nabla v|^2\,\mathrm{d}x + \frac{4(CK_\alpha+1)}{(\overline{t}-t_0)^2} \int_{D(t_0,\overline{z})} |x-\overline{x}|^{\alpha} v^2\,\mathrm{d}x.$$

By means of Hölder inequality we have

$$\frac{1}{(\overline{t}-t_0)^2} \int_{D(t_0,\overline{z})} |x-\overline{x}|^{\alpha} v^2\,\mathrm{d}x \leq \left(\frac{4\pi}{3}\right)^{2/3} \left(\int_{D(t_0,\overline{z})} |x-\overline{x}|^{3\alpha} v^6\,\mathrm{d}x\right)^{1/3}.$$

Combining these relations with (2.7) we obtain

$$\int_{D(t_0,\overline{z})} |x-\overline{x}|^{\alpha-2} v^2\,\mathrm{d}x \leq 2K_\alpha(\overline{t}-t_0)^{\alpha} E_0 + C_\alpha(\overline{t}-t_0)^{\frac{2\alpha}{3}} E_0^{1/3};$$

here $C_\alpha = 4(CK_\alpha+1)\sqrt[3]{4\pi^2/3}$. Coming back to (3.9), this implies

$$\int_{M_{t_0}^t(\overline{z})} |\xi-\overline{x}|^{\alpha-1} u^4\,\mathrm{d}\omega \leq 2\sqrt{3}[\sqrt{2K_\alpha}E_0 + \sqrt{C_\alpha}E_0^{2/3}].$$

To conclude the proof we determine $\varepsilon_0(\alpha)$ such that this quantity is not greater than 2π. This means

$$\varepsilon_0(\alpha) = \max\left\{\frac{\pi}{2\sqrt{6K_\alpha}}, \left(\frac{\pi}{2\sqrt{3C_\alpha}}\right)^{3/2}\right\} = \frac{\pi}{2\sqrt{6K_\alpha}} = \frac{\pi(\alpha+1)}{4\sqrt{6}}.$$

In particular $\varepsilon_0(\alpha)$ is an increasing function in α. From $E_0 \leq \varepsilon_0(\alpha)$ we get (3.8) and then the conclusion. The regularity argument is standard. $\square$

The fact that $\varepsilon_0(\alpha)$ is an increasing function in α, means that the vanishing potentials helps the global existence, despite of the technical difficulty to treat this case.

In particular for $\alpha = 0$ and $V(x) = \lambda > 0$ Rauch's result gives $\varepsilon_0 = \pi/\sqrt{2\lambda}$; in some sense a vanishing potential having high order behaves like a small constant λ.

4. Other type of potentials. The main ingredients of the previous proof are Kirchhoff formula and Hardy inequality. We can choose also the potential in a wider class so that the problem

$$\Box\, u(x,t) = -V(x)u^5 \tag{4.1}$$

$$\begin{aligned} u(0,x) &= f(x) \\ u_t(0,x) &= g(x) \end{aligned} \tag{4.2}$$

has global solution.

We shall see that a result in this direction is related to the order of the zeros of the potential. We say that $\overline{x}$ is a zero of order m for $V(x)$ if $V(\overline{x}) = 0$ and all derivaties of $V(x)$ up to the order m vanish in $\overline{x}$.

In the first part of this section, we consider $V \in \mathcal{C}^2(\mathbb{R}^3)$, $V(x) \geq 0$ having a unique zero $\overline{x}$. At the begin we try to apply the same argument of the previous proof taking $u = \sqrt{V}u$ and $\alpha = 0$. We have to establish

$$\int_{t_0}^{t} (t-\sigma)^{-1} \int_{|x-\xi|=t-\sigma} V(\xi)u^4(\xi,\sigma)\,\mathrm{d}\xi\,\mathrm{d}\sigma \leq 2\pi \tag{4.3}$$

for all $(x,t) \in K_{t_0}(\overline{z}) \setminus \{\overline{z}\}$, being $\overline{z} = (\overline{x}, \overline{t})$.
We choose t_0 such that in each section of $K_{t_0}(\overline{z})$ we have $V(\overline{x}) \leq 1$. The difficulty arises from the estimate of $\displaystyle\int_{D(t_0,\overline{z})} V(x)\frac{u^2}{|x-\overline{x}|^2}\,\mathrm{d}x$. We find

$$\begin{aligned} &\int_{D(t_0,\overline{z})} V(x)\frac{u^2}{|x-\overline{x}|^2}\,\mathrm{d}x \leq \\ &\leq C \sup_{x\in D(t_0,\overline{z})} V^{2/3}(x) \left[\int_{D(t_0,\overline{z})} |\nabla u|^2\,\mathrm{d}x + \left(\int_{D(t_0,\overline{z})} V(x)u^6\,\mathrm{d}x \right)^{1/3} \right] + \\ &+C \int_{D(t_0,\overline{z})} \left|\nabla\sqrt{V(x)}\right|^2 u^2\,\mathrm{d}x. \end{aligned} \tag{4.4}$$

In order to control the last term with the initial energy, we have to suppose that

$$\left|\nabla\sqrt{V}\right|^2 \leq C\sqrt[3]{V} \tag{4.5}$$

in $D(t_0,\overline{z})$. In force of our choice of t_0 we see that $\displaystyle\sup_{x\in D(t_0,\overline{z})} V^{2/3}(x) \leq 1$. Hence we find ε_0, such that if $E_0 \leq \varepsilon_0$ the problem (2.1), (2.2) has classical solution. In particular ε_0 can be taken uniform with respect to all V which satisfy (4.5).

Example. Let $n_1, n_2, n_3 \geq 2$ be positive integers. The following potential satisfies (4.5):

$$V(x_1, x_2, x_3) = x_1^{2n_1} + x_2^{2n_2} + x_3^{2n_3}.$$

In this example the constant in (4.5) does not depend on the derivaties of V. If we remove this assumption a more clear condition on V can be given. We prove that (4.5) holds for each $V(x)$ positive function such that

$$\begin{aligned} &V^\alpha(x) \in \mathcal{C}^2(D(t_0, \overline{z})) \quad 0 \le \alpha \le 2/3 \\ &\overline{x} \text{ is a zero of order } 2. \end{aligned} \tag{4.6}$$

In order to prove that this condition implies (4.5), we use a simple result due to Glaeser. In [1] the following relations are shown:

(i) For any $f \in \mathcal{C}^2(\mathbb{R})$, $f \ge 0$ such that $f'' \le M$ one has $|f'| \le 2Mf$.

(ii) Let $\overline{x}$ be a zero of second order for $f \in \mathcal{C}^2(\mathbb{R})$, $f \ge 0$. Then $\sqrt{f}$ is a $\mathcal{C}^1$ function.

It is possible to derive the n-dimensional and local variant of these assertions. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $F : \Omega \to \mathbb{R}$ be a $\mathcal{C}^2$ positive function having all zeros of second order. There exists a compact $K \supset \Omega$ such that for all $x \in \Omega$

$$\left|\nabla\sqrt{F(x)}\right|^2 \le \frac{n}{2} \max_{x\in K} \max_{i,j} |\partial_{x_i,x_j} F(x)|.$$

Applying this inequality to $V^\alpha(x)$, we have (4.5) under the condition $0 \le \alpha \le 2/3$. In particular the previous example satisfies (4.6). We underline that in this case $\varepsilon_0(V)$ depends on the second derivaties of V in $D(t_0, \overline{z})$.

The next problem is to treat with a potential V which vanishes with low order than the one required from the condition (4.6).
A first idea is to use Hölder inequality in (4.4) and then to control the last term of this computation by means of the initial energy. We have to require

$$\frac{|\nabla V|^3}{V^2} \in L^1(D(t_0, \overline{z})). \tag{4.7}$$

Another possibility is to apply directly an inequality of the following type:

$$\int_{\mathbb{R}^3} V(x) \frac{\omega^2}{|x - \overline{x}|^2} \,\mathrm{d}x \le C_V \int_{\mathbb{R}^3} V(x) |\nabla\omega|^2 \,\mathrm{d}x \quad \forall \omega \in \mathcal{C}_0^1(\mathbb{R}^3). \tag{4.8}$$

This weighted Hardy inequality is well investigated in literature and some sufficient conditions on the weight are given in term of differential equations. For example we can choose a potential $V(x)$ defined by the aid of a $\mathcal{C}^1$ function

$F : \mathbb{R}^3 \to \mathbb{R}^3$, $F = (f_1, f_2, f_3)$, such that $\text{div}F > 0$. The inequality (4.8) holds for all $V(x)$ positive functions satisfying

$$\begin{cases} V(x) = |x - \overline{x}|^2 \text{div}F \\ \\ |f_i| \le C|x - \overline{x}|\text{div}F \quad i = 1, 2, 3 \end{cases} . \tag{4.9}$$

This fact can be deduced from the inequalities exploited by Opic (cf. [2, Theorem 14.9]). Using (4.8) instead of (3.4), the same computations of the proof of Theorem 3.1 allow to have existence of the global solution for the Problem (4.1) (4.2) provided $V(\overline{x}) = 0$ and $V(x)$ satisfies (4.9) in a neighborhood of $\overline{x}$. In this case we also require $E_0 \le \varepsilon_0(V)$.

Example. We consider $F = (x_1 \ln(x_1^2 + 1), x_2, x_1)$ and

$$V(x_1, x_2, x_3) = |x|^2 \left(\ln(x_1^2 + 1) + \frac{2x_1^2}{x_1^2 + 1} + 1 \right).$$

The functions F, V verify (4.9).

Finally, it is possible to obtain global existence for (4.1), (4.2) with small energy, if the potential vanishes in different points. We set

$$Z(V) = \{x \in \mathbb{R}^3 \mid V(x) = 0\}.$$

Suppose that for each $\overline{x} \in Z(V)$, either $V(x) = |x - \overline{x}|^\alpha$ or one of the conditions (4.5), (4.6), (4.7), (4.9) are fulfilled in a neighborhood of $\overline{x}$. Form the previous discussion it follows that for any $\overline{x} \in Z(V)$ there exists $\varepsilon_0(V, \overline{x})$ such that if $E_0 \le \varepsilon_0(V, \overline{x})$, $\overline{x}$ is not a blow up point.
If $Z(V)$ is compact we have directly global existence if $E_0 \le \varepsilon_0$ with

$$\varepsilon_0 = \min_{x \in Z(V)} \{\varepsilon_0(V, x)\}.$$

Let us consider the case in which $Z(V)$ is not compact.
If for each $\overline{x} \in Z(V)$ there exists a neighborhood in which $V(x) = |x - \overline{x}|^\alpha$ or $V(x)$ satisfies (4.5), then we get the global existence result whatever is the distribution of the zeros of the potential. This can be proved fixing for all $\overline{x} \in Z(V)$ a suitable t_0 such that $V(x) \le 1$ in $D(t_0, \overline{z})$.
If $\overline{x} \in Z(V)$ implies $V(x) = |x - \overline{x}|^\alpha$ or $V(x)$ satisfies (4.5) or (4.6) in a neighborhood of $\overline{x}$, then $\varepsilon_0(V, \overline{x})$ depends on $\sup_{D(t_0, \overline{z})} |D^2 V(x)|$. Suppose $V(x)$ has all the second derivaties bounded, the quantity $\varepsilon_0(V, \overline{x})$ does not depend on $\overline{x}$ and we get global existence result without other hypotheses on $Z(V)$.

To conclude this work we give a sort of converse of these results.
Let be $u : \mathbb{R}^3 \times [0, T[\to \mathbb{R}$ a local solution of (4.1), (4.2), with $V(x)$ having

admissible behaviour, this means that the previous conditions on V and $Z(V)$ are fulfilled.
It is clear that if $E_0 \leq \varepsilon_0(V)$ then $E(u, D(t, \overline{z})) \leq \varepsilon_0(V)$ for all $t < \overline{t}$.
On the contrary if u blows up, $E_0 > \varepsilon_0(V)$. Since the solution is classical until the blow up point, one can expect that $E(u, D(t, \overline{z})) > \varepsilon_0(V)$ for all $t \in [0, \overline{t}[$. This is established in the next theorem.

Theorem 4.1. *Let us consider the Problem* (4.1), (4.2), *with* $V(x) \geq 0$ *a* $\mathcal{C}^2(\mathbb{R}^3)$ *function. Suppose there exists* $\varepsilon_0(V)$ *such that if* $E_0 \leq \varepsilon_0(V)$ *the problem has global existence.*
If $\overline{z} = (\overline{x}, \overline{t})$ *is a blow up point, that is* $\lim_{z \to \overline{z}} \left| u_{|K_0(\overline{z})}(z) \right| = +\infty$, *then for all* $t \in [0, \overline{t}[$ *we have* $E(u, D(t, \overline{z})) > \varepsilon_0(V)$.

P r o o f. We argue by contradiction. Let $t_0 \in [0, \overline{t}[$ such that $E(u, D(t_0, \overline{z})) \leq \varepsilon_0(V)$. We can choose a sequence $\{\delta_m\}$ such that $t_0 \leq \delta_m \leq \overline{t}$, $\delta_m \to \overline{t}$ and $\sup_{K_{t_0}^{\delta_m}} |u(x,t)| \to \infty$. From Kirchhoff formula we have

$$\sup_{K_{t_0}^{\delta_m}} |u(x,t)| \leq \sup_{K_{t_0}^{\delta_m}} \left\{ |\underline{u}(x,t)| + \frac{1}{4\pi} \int_{M_{t_0}^t(\overline{z})} \frac{V(\xi)}{|\xi - \overline{x}|} u^5(\xi, \sigma)\, \mathrm{d}\omega \right\}. \tag{4.10}$$

We shall show

$$\int_{M_{t_0}^t(\overline{z})} \frac{V(\xi)}{|\xi - \overline{x}|} u^5(\xi, \sigma)\, \mathrm{d}\omega \leq C \sup_{D(t_0, \overline{z})} |V(\xi)| \frac{E(u, D(t_0, \overline{z}))}{|t|}; \tag{4.11}$$

hence from $E(u, D(t_0, \overline{z})) \leq \varepsilon_0(V)$ we conclude that the right side of (4.10) is bounded while the left side goes to infinity when $\delta_m \to T$. This yields the conclusion.
It remains to check (4.11). We observe that

$$\int_{M_{t_0}^t(\overline{z})} \frac{V(\xi)}{|\xi - \overline{x}|} u^5\, \mathrm{d}\omega \leq \sqrt{2} \int_{D(t_0, \overline{z}) \setminus D(t, \overline{z})} \frac{V(\xi)}{|\xi - \overline{x}|} u^5(\xi, t - |\xi - \overline{x}|)\, \mathrm{d}x \leq$$

$$\leq \sqrt{2} \| |x - \overline{x}|^{-1} \|_{L^6(D(t_0, \overline{z}) \setminus D(t, \overline{z}))} \left(\int_{D(t_0, \overline{z})} V(x)^{6/5} u^6\, \mathrm{d}x \right)^{5/6}.$$

Using polar coordinates we obtain $\| |x - \overline{x}|^{-1} \|_{L^6(D(t_0, \overline{z}) \setminus D(t, \overline{z}))} \leq C/|t|$. Hence we find (4.11). $\square$

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